

Assumed Knowledge Integrals of simple functions such as x^n (including $1/x$), $\sin x$, $\cos x$, e^x . The simple properties of definite integrals.

Objectives

- (2a) To be able to estimate the definite integral of a function and the error of the estimate using Riemann sums.
- (2b) To be able to estimate a sum of a series by comparing it with a definite integral.
- (2c) To understand that integration and differentiation are inverse operations, i.e. that the definite integral of the derivative of a function is equal to the difference between the function values at the end-points, $\int_a^b F'(x) dx = F(b) - F(a)$.
- (2d) To understand the conditions under which the definite integral of a function exists.
- (2e) To be able to use the properties of definite integrals confidently.

Preparatory Questions

1. Let f and g be continuous functions on the interval $[a, b]$, and let $a \leq c \leq b$.
 - (i) If $\int_a^b f(x)dx = 10$ and $\int_c^b 2f(x)dx = 5$, find $\int_a^c f(x)dx$.
 - (ii) If $\int_a^b f(x)dx = 12$ and $\int_a^b (2f(x) + 3g(x))dx = 63$, find $\int_a^b g(x)dx$.
 - (iii) If $\int_c^a f(x)dx = 5$ and $\int_b^c f(x)dx = -2$, find $\int_a^b 2f(x)dx$.
2. A MATH1003 student reasons that because $-1/x$ has derivative $1/x^2$ the Fundamental Theorem of Calculus gives:

$$\int_{-1}^1 \frac{1}{x^2} dx = \left[-\frac{1}{x} \right]_{-1}^1 = -2.$$

Her friend says: “But $\frac{1}{x^2}$ is always positive, so its graph is above the x -axis. So surely, the integral is a positive number.”

Which argument do you find more persuasive?

Practice Questions

3. Sort out the apparent contradiction in Question 2.

4. (i) Use the Fundamental Theorem to find $\int_1^a \frac{1}{x^2} dx$ (where a is any real number greater than 1).
- (ii) What is $\lim_{a \rightarrow \infty} \int_1^a \frac{1}{x^2} dx$?
- (iii) Draw a sketch of the function $\frac{1}{x^2}$ and use it to interpret your answer to part (ii) geometrically.
- (iv) Find a lower Riemann sum for the function $\frac{1}{x^2}$ on the interval $[1, \infty)$, using sub-intervals of width 1.
- (v) Use parts (ii) and (iv) to show that $\sum_{n=1}^{\infty} \frac{1}{n^2} < 2$.
- (vi) Calculate $\sum_{n=1}^7 \frac{1}{n^2}$. (vii) Deduce that $1.5 < \sum_{n=1}^{\infty} \frac{1}{n^2} < 2$.
5. (Suitable for group work and discussion) Use an appropriate integral, and Riemann sums, to find an estimate for $1 + \sqrt{2} + \sqrt{3} + \dots + \sqrt{100}$.

More Questions

6. Consider a partition of $[0, 1.75]$ into 7 equal parts. Given the values of $\cos(x^2)$ tabulated below, use lower and upper Riemann sums to find lower and upper estimates for $\int_0^{1.75} \cos(x^2) dx$.

x	0	0.25	0.5	0.75	1	1.25	1.5	1.75
$\cos(x^2)$	1	0.998	0.969	0.846	0.54	0.008	-0.628	-0.997

Remember that x should be measured in radians.

7. (i) Consider the partition of $[0, 1]$ into five equal parts. Use lower and upper Riemann sums to find lower and upper bounds for $\int_0^1 x^5 dx$.
- (ii) Obtain lower and upper bounds for $\sum_{i=1}^{25} i^4$.

Answers to Selected Questions

1. (i) 7.5 (ii) 13 (iii) -6
4. (i) $-\frac{1}{a} + 1$ (ii) 1 (iv) $\sum_{n=2}^{\infty} \frac{1}{n^2}$ (v) 1.511...
5. $\frac{2000}{3} < 1 + \sqrt{2} + \sqrt{3} + \dots + \sqrt{99} + \sqrt{100} < \frac{2030}{3}$.
6. $0.434 \leq \int_0^{1.75} \cos(x^2) dx \leq 0.93325$.
7. (i) $0.0832 \leq \int_0^1 x^5 dx \leq 0.2832$. (ii) $1953125 \leq \sum_{i=1}^{25} i^4 \leq 2343750$.