The University of Sydney
Math1003 Integral Calculus and Modelling

Semester 2 **Exercises for Week 4** 2011

Assumed Knowledge Sketching curves of simple functions. Integrals of simple functions such as x^n (including $1/x$), $\sin x$, $\cos x$, e^x.

Objectives

(3a) To be able to interpret an area or volume as the limit of an appropriate Riemann sum.

(3b) To be able to use elementary slices to construct Riemann sums for areas and volumes.

(3c) To be able to use elementary shells to construct Riemann sums for volumes.

(3d) To understand the substitution formula and be able to use simple substitutions to evaluate definite integrals.

Preparatory Questions

1. Evaluate the following definite integrals:

(i) $\displaystyle\int_0^{5\pi} \cos\left(\frac{x}{10}\right) dx.$ (ii) $\displaystyle\int_1^2 \sqrt{x-1}\, dx.$ (iii) $\displaystyle\int_0^1 x^\pi\, dx.$

2. Sketch the region bounded by the curves $y = \sin x$ and $y = \sin 2x$, and the straight lines $x = 0$ and $x = \pi/2$.

3. Sketch the region of the xy-plane bounded by the x-axis, the line $x = 2$, and the graph of $y = x$. This region is rotated about the line $x = 4$. Sketch an elementary disk that you would use to construct the volume.

Practice Questions

4. Evaluate the following definite integrals by making a substitution.

(i) $\displaystyle\int_0^1 \frac{x^2}{\sqrt{2+x^3}} dx.$ (ii) $\displaystyle\int_0^1 (2x+1)(x^2+x+1)^3 dx.$

(iii) $\displaystyle\int_0^{\pi/2} \cos^3 x\, dx.$ Hint: First use the identity $\cos^2 x = 1 - \sin^2 x$.

5. Find the area of the region sketched in Question 2.

6. Calculate the volume generated in Question 3.

7. Suppose that a bagel cut horizontally in half has the shape given by rotating, about the y axis, the area bounded by the curve $y = 3x - x^2 - 2$ and the x-axis.

(i) Sketch the curve $y = 3x - x^2 - 2$.

(ii) Consider a vertical strip under the graph at some point x $(1 \leq x \leq 2)$, of width Δx, which is rotated about the y axis to give a cylindrical shell. Sketch the shell.

(iii) Imagine the shell being cut vertically and opened out flat. Thus find the volume of the cylindrical shell.

(iv) Write down the volume of the half bagel as a definite integral.

(v) Evaluate the definite integral to find the volume of the half bagel.

More Questions

8. Evaluate the following definite integrals using the substitution $u = \sec x$.

(i) $\displaystyle\int_0^{\pi/4} \tan x \sec^3 x\, dx.$ (ii) $\displaystyle\int_0^{\pi/3} \sec^5 x \tan^3 x\, dx.$

Hint: Use a trigonometric identity in (ii).

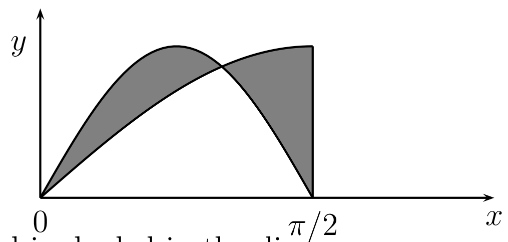
9. Evaluate the following definite integrals using a substitution.

(i) $\displaystyle\int_0^{1/2} \frac{x}{\sqrt{1-x^2}} dx.$ (ii) $\displaystyle\int_0^{\pi/2} \sin^3 x \cos^4 x\, dx.$

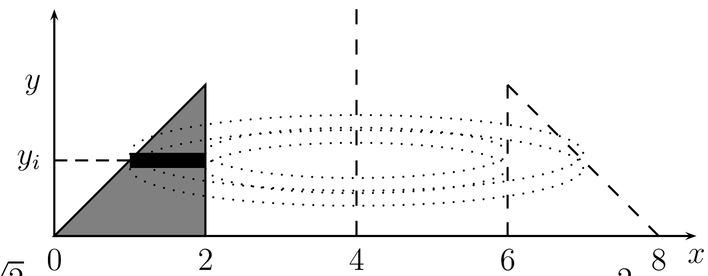
Answers to Selected Questions

1. (i) 10. (ii) $\frac{2}{3}$. (iii) $\frac{1}{\pi+1}$.

2. The curves intersect when $\sin x = \sin 2x$, i.e., when $\sin x = 2\sin x \cos x$ or $\sin x(1 - 2\cos x) = 0$. The roots occur when $\sin x = 0$ and $\cos x = 1/2$. For $0 \leq x \leq \pi/2$, $\sin x = 0$ when $x = 0$ and $\cos x = 1/2$ when $x = \pi/3$.



3. The region to be rotated is shaded in the diagram.



4. (i) $\frac{2\sqrt{3}}{3} - \frac{2\sqrt{2}}{3}$ (ii) 20 (iii) $\frac{2}{3}$

5. $\frac{1}{2}$

6. $\frac{32\pi}{3}$

7. (iii) $2\pi(3x^2 - x^3 - 2x)\Delta x$ (iv) $\displaystyle\int_1^2 2\pi(3x^2 - x^3 - 2x)dx$ (v) $\frac{\pi}{2}$

8. (i) $\frac{2\sqrt{2}}{3} - \frac{1}{3}$ (ii) $\frac{418}{35}$

9. (i) $-\frac{\sqrt{3}}{2} + 1$ (ii) $\frac{2}{35}$