

Assumed Knowledge Sketching curves of simple functions. Integrals of simple functions such as x^n (including $1/x$), $\sin x$, $\cos x$, e^x .

Objectives

- (4a) To understand and be able to use integration by parts to evaluate definite integrals.
- (4b) To understand that an indefinite integral is a function.
- (4c) To understand that differentiation and (indefinite) integration are inverse processes when applied to functions.
- (4d) To be able to sketch a function defined by an integral.
- (4e) To be able to derive a reduction formula for an integral.

Preparatory Questions

1. (i) Find the indefinite integral $\int \tan x \, dx$.

Hint: Use $\tan x = \frac{\sin x}{\cos x}$ and substitute $u = \cos x$.

- (ii) Use two integration by part to find the indefinite integral $\int x^2 e^x \, dx$.

Practice Questions

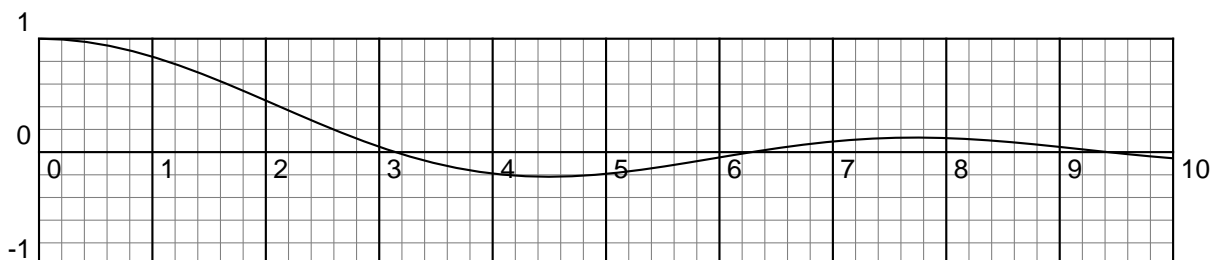
2. Evaluate the following integrals by using integration by parts.

(i) $\int_0^{1/2} x e^{2x} \, dx$. (ii) $\int_0^{\pi/4} \theta \sin 4\theta \, d\theta$. (iii) $\int_1^2 t^2 \ln t \, dt$.

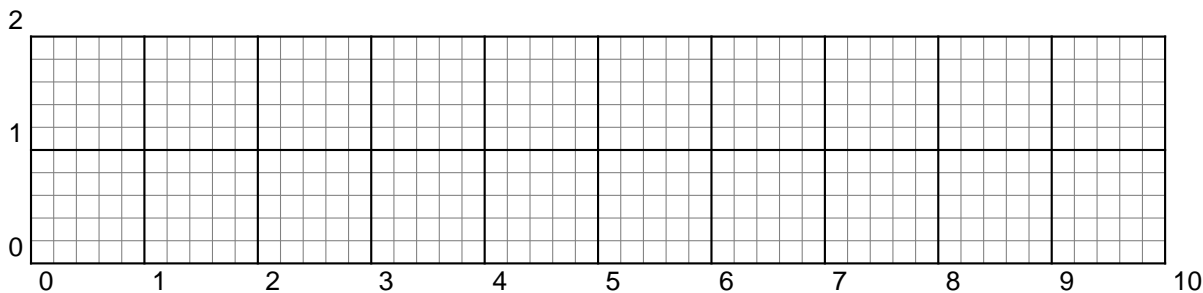
3. Define $\text{Si}(x)$ as $\text{Si}(x) = \int_0^x f(t) \, dt$, where $f(t) = \begin{cases} \frac{\sin t}{t} & t \neq 0 \\ 1 & t = 0. \end{cases}$

This function is called the *sine-integral*, and is useful in optics.

This is the graph of $f(t)$.



- (i) What is $\text{Si}'(x)$?
- (ii) What is the value of $\text{Si}(0)$?
- (iii) For $0 \leq x \leq 3\pi$, use the graph of $f(t)$ to determine the values of x for which $\text{Si}(x)$ is increasing, and the values of x for which it is decreasing.
- (iv) For which values of x between 0 and 3π does $\text{Si}(x)$ have stationary points?
- (v) Use the graph of $f(t)$ to estimate $\text{Si}(\pi)$, $\text{Si}(2\pi)$ and $\text{Si}(3\pi)$.
- (vi) Use the graph of $f(t)$ to determine values of x at which Si has points of inflection.
- (vii) Sketch the graph of Si for $0 \leq x \leq 3\pi$.



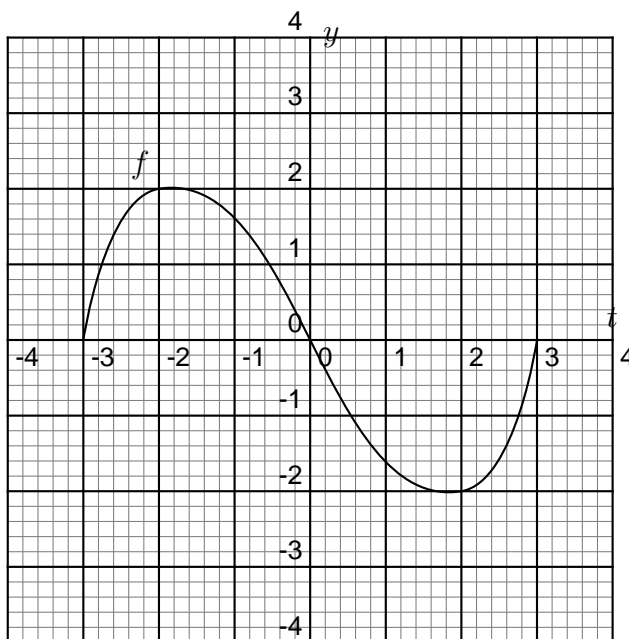
4. Establish the following reduction formula. [Hint: Write the integrand as $u(x)v'(x)$ where $u = \cos^{n-1} x$.]

$$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx .$$

Use this formula to find $\int \cos^2 x \, dx$ and $\int \cos^4 x \, dx$.

More Questions

5. Let $g(x) = \int_{-3}^x f(t) \, dt$ where f is the *odd* function whose graph is shown.



- (i) Evaluate $g(-3)$ and $g(3)$.
(ii) Estimate $g(-2)$, $g(-1)$ and $g(0)$.
(iii) On what interval is g increasing?
(iv) Where does g have a maximum value?
(v) Sketch a rough graph of g .
6. Evaluate the following integrals by using integration by parts.
- (i) $\int_0^1 (2x + 3)e^x dx$. (ii) $\int_0^\pi \theta^2 \cos 3\theta d\theta$. (iii) $\int_{-\pi/4}^{\pi/4} t \sin t \cos t dt$.

Hint: First use an identity in (iii).

7. Let $I_n = \int x^n e^x dx$. Use integration by parts to establish the reduction formula

$$I_n = x^n e^x - nI_{n-1}.$$

Hence find $\int x^3 e^x dx$.

8. (i) Let $I_n = \int x(\ln x)^n dx$. Use integration by parts to establish the reduction formula

$$I_n = \frac{1}{2}x^2(\ln x)^n - \frac{n}{2}I_{n-1}.$$

- (ii) Starting with $I_0 = \int x dx = \frac{1}{2}x^2 + C$, use the reduction formula from part (i) to find I_2 .

Answers to Selected Questions

1. (i) $-\ln |\cos x| + C$. (ii) $(x^2 - 2x + 2)e^x + C$.
2. (i) $\frac{1}{4}$. (ii) $\frac{\pi}{16}$. (iii) $\frac{8}{3} \ln 2 - \frac{7}{9}$.
3. (iii) $\text{Si}(x)$ is increasing for $0 < x < \pi$ and $2\pi < x < 3\pi$, and decreasing for $\pi < x < 2\pi$.
(iv) π , 2π and 3π
(v) $\text{Si}(\pi) \approx 1.9$, $\text{Si}(2\pi) \approx 1.4$, $\text{Si}(3\pi) \approx 1.7$
(vi) 0, 4.5, 7.7
4. $I_2 = \frac{1}{2} \cos x \sin x + \frac{1}{2}x + C$
5. (i) $g(-3) = g(3) = 0$ (ii) $g(-2) \approx 1.4$, $g(-1) \approx 3.3$, $g(0) \approx 4.2$ (iii) $(-3, 0)$ (iv) $t = 0$
6. (i) $3e - 1$. (ii) $-\frac{2\pi}{9}$. (iii) $\frac{1}{4}$.
7. $I_3 = x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C$
8. $I_2 = \frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2 \ln x + \frac{1}{4}x^2 + C$