The University of Sydney
Math1003 Integral Calculus and Modelling

Semester 2 **Exercises for Week 6** 2011

Assumed Knowledge Simple properties of the functions $\ln x$ and e^x, including their derivatives.

Objectives

(5a) To know and be able to use the properties of the ln function.

(5b) To know and be able to use the properties of the exp function.

(5c) To know and be able to use the properties of the generalised exponential function a^x.

(5d) To be able to perform a logarithmic differentiation.

Preparatory Questions

1. Simplify each of the following expressions:

(i) $e^{\ln 6}$ (ii) $\ln \sqrt{e}$ (iii) $e^{x+\ln x}$.

2. Find dy/dx for each of the following:

(i) $y = 2^x$ (ii) $y = \ln\left(\dfrac{1+x}{1-x}\right)$

(iii) $y = \log_2 x$ (iv) $y = \log_{10} x$

Practice Questions

3. Find dy/dx for each of the following:

(i) $y = 3\log_2(x^2)$ (ii) $y = \log_{10}\sqrt{x}$

(iii) $y = x^x$ (iv) $y = (\sin x)^x$

(v) $y = \dfrac{x^2\sqrt[3]{7x-4}}{(1+x^2)^4}$

4. Recall the hyperbolic sine and hyperbolic cosine functions, $\sinh x$ and $\cosh x$, are defined as $\sinh x = \frac{1}{2}(e^x - e^{-x})$ and $\cosh x = \frac{1}{2}(e^x + e^{-x})$.
Using properties of the exponential function, show that

(i) $\dfrac{d}{dx}\sinh x = \cosh x$ (ii) $\dfrac{d}{dx}\cosh x = \sinh x$.

(iii) $\cosh A \cosh B + \sinh A \sinh B = \cosh(A+B)$

(iv) $2(\cosh A)^2 - 1 = \cosh(2A)$.

5. (*i*) Use l'Hôpital's rule to find $\lim_{x\to 0} \ln\{(1+x)^{1/x}\}$.

(*ii*) Hence find $\lim_{x\to 0}(1+x)^{1/x}$.

[Note: $\lim_{x\to 0} \ln\{(1+x)^{1/x}\} = \ln\left(\lim_{x\to 0}(1+x)^{1/x}\right)$. Can you say why?]

6. (*i*) For which positive real numbers x is it true that $\sqrt{x} > x/2$?

(*ii*) *Without* using your calculator, and assuming that $\pi \approx 3$, determine which of the following is bigger: $(\sqrt{\pi})^{\pi}$ or $\pi^{\sqrt{\pi}}$.
[Hint: The exponential function is always increasing, so if $a > b$, then $e^a > e^b$.]

More Questions

7. Consider the function $f(x) = \dfrac{\ln x}{x}$, which is defined for all $x > 0$. Show that this function is strictly increasing on the interval $(0, e)$, strictly decreasing on the interval (e, ∞), and thus has a global maximum at $x = e$. Hence show that $f(x) \le \dfrac{1}{e}$ for all $x > 0$. Use this result with $x = \pi$ to show that $\pi^e < e^{\pi}$.

Answers to Selected Questions

1. (*i*) $e^{\ln 6} = 6$. (*ii*) $\ln\sqrt{e} = \ln(e^{1/2}) = \frac{1}{2}\ln e = \frac{1}{2}$.

(*iii*) $e^{x+\ln x} = e^x e^{\ln x} = xe^x$.

2. (*i*) $2^x \ln 2$ (*ii*) $\dfrac{2}{1-x^2}$

(*iii*) $\dfrac{1}{x\ln 2}$ (*iv*) $\dfrac{1}{x\ln 10}$

3. (*i*) $\dfrac{6}{x\ln 2}$ (*ii*) $\dfrac{1}{2x\ln 10}$ (*iii*) $x^x(1+\ln x)$

(*iv*) $(\sin x)^x(x\cot x + \ln(\sin x))$ (*v*) $\dfrac{x^2\sqrt[3]{7x-4}}{(1+x^2)^4}\left(\dfrac{2}{x} + \dfrac{7}{3(7x-4)} - \dfrac{8x}{1+x^2}\right)$

5. (*i*) 1 (*ii*) e

6. (*i*) $0 < x < 4$ (*ii*) $(\sqrt{\pi})^{\pi} < \pi^{\sqrt{\pi}}$