

Assumed Knowledge Integration techniques.

Objectives

(8a) To be able to write a rational expression as a sum of partial fractions.

(8b) To be able to solve a separable equation using partial fractions.

Preparatory Questions

1. Use partial fractions to express the following as a sum of terms:

(i) $\frac{1}{(x+1)(x-2)}$

(ii) $\frac{7}{2x^2+5x-3}$

(iii) $\frac{1}{(x+a)(x+b)}$

(iv) $\frac{x}{x-5}$

Practice Questions

2. Evaluate the integrals:

(i) $\int \frac{1}{(x+1)(x-2)} dx$

(ii) $\int \frac{7}{2x^2+5x-3} dx$

(iii) $\int \frac{1}{(x+a)(x+b)} dx$

(iv) $\int \frac{x}{x-5} dx$

3. (Suitable for group work and discussion.) A trout farmer believes that, if she doesn't remove any fish from a certain tank, the population of fish is modelled by the differential equation

$$\frac{dP}{dt} = 2P - 0.01P^2,$$

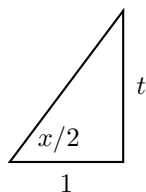
where P is the number of fish after t years.

- (i) Suppose the farmer initially stocks the tank with 50 fish.
 - (a) What is the maximum number of fish the tank can support?
 - (b) Find P for which the population is increasing most rapidly.
 - (c) Find an explicit formula for P in terms of t .
 - (d) How long will it be before there are 199 fish in the tank?
 - (e) Sketch a graph of $y = P(t)$.

- (ii) Now suppose that the farmer initially stocks the tank with 250 fish. Find the solution $P(t)$, and sketch its graph.
- (iii) If the trout farmer were to remove 75 fish from the tank each year, then the differential equation modelling population size would be

$$\frac{dP}{dt} = 2P - 0.01P^2 - 75.$$

- (a) Find a solution to this differential equation, supposing that the farmer initially stocks the tank with 60 fish.
- (b) According to this model, what is the size of the fish population in the long-term?
- (c) What should the farmer expect to happen if she initially stocks the tank with 50 fish?
4. (i) Use the geometry of the following right-angled triangle in which $\tan(x/2) = t$



to show that

(a) $\sin \frac{x}{2} = \frac{t}{\sqrt{1+t^2}}$

(b) $\cos \frac{x}{2} = \frac{1}{\sqrt{1+t^2}}$.

(ii) Hence show that

(a) $\cos x = \frac{1-t^2}{1+t^2}$

(b) $\sin x = \frac{2t}{1+t^2}$.

(iii) If $t = \tan \frac{x}{2}$, show that $\frac{dx}{dt} = \frac{2}{1+t^2}$.

(iv) Use the substitution $t = \tan \frac{x}{2}$ to evaluate

$$\int \frac{5}{4 \sin x + 3 \cos x} dx.$$

More Questions

5. (i) An algal bloom will occur in water polluted by excess nutrients when the number of algae cells x increases dramatically. The rate of increase of the number of algae cells in a sample of such water during a bloom is proportional to the number of cells present at any instant. Write down a differential equation that models this phenomenon.
- (ii) If, however, the water is low in oxygen then the rate of increase is proportional not to the total number of algae cells x but instead, at any time t ($t \geq 0$), to $e^{-at}x$, where a is a positive constant. [That is, the rate of increase is proportional to a fraction of the total number of cells, and that fraction decreases exponentially with time.]

- (a) Write down a differential equation that expresses these statements.
 (b) Solve it to obtain the number of cells in a sample as a function of time, given the initial number of algae cells.
 (c) Is there a limit to the number of cells as $t \rightarrow \infty$?

Answers to Selected Questions

1. (i) $\frac{1}{3} \left(\frac{1}{x-2} - \frac{1}{x+1} \right)$ (ii) $\frac{2}{2x-1} - \frac{1}{x+3}$
 (iii) $\frac{1}{b-a} \left(\frac{1}{x+a} - \frac{1}{x+b} \right)$ (iv) $1 + \frac{5}{x-5}$
2. (i) $\frac{1}{3} \ln \left| \frac{x-2}{x+1} \right| + C$ (ii) $\ln \left| \frac{2x-1}{x+3} \right| + C$
 (iii) $\frac{1}{b-a} \ln \left| \frac{x+a}{x+b} \right| + C$ (iv) $x + 5 \ln |x-5| + C$
3. (i) (a) 200 fish
 (b) $P = 100$
 (c) $P(t) = \frac{200}{1 + 3e^{-2t}}$
 (d) Just over 3 years
 (ii) $P = \frac{200}{1 - \frac{1}{5}e^{-2t}}$
 (iii) (a) $P = \frac{150(1 + 3e^{-t})}{1 + 9e^{-t}}$
 (b) $P \rightarrow 150$
4. (iv) $\ln \left| \frac{3 \tan(x/2) + 1}{\tan(x/2) - 3} \right| + C$
5. (ii) (a) $dx/dt = ke^{-at}x$
 (b) $x = x_0 \exp \left[\left(\frac{k}{a} \right) (1 - e^{-at}) \right]$