The University of Sydney

Math1003 Integral Calculus and Modelling

Semester 2 **Exercises for Week 10** 2011

Assumed Knowledge Integration techniques.

Objectives

(9a) To be able to distinguish between separable and linear first-order differential equations.

(9b) To be able to solve a linear equations utilising an integrating factor.

Preparatory Questions

1. For each of the following equations, determine whether it is separable or a first-order linear equation:

(i) $\dfrac{dy}{dx} + 3y = x$ (ii) $t\dfrac{dx}{dt} + x = \cos t$

(iii) $\dfrac{x}{2}\dfrac{dy}{dx} = x^2 - y$ (iv) $\dfrac{dy}{dx} = \dfrac{2x\sqrt{y}}{\sqrt{1+x^2}}$.

2. Write each of the linear equations in Question 1 in standard form $\dfrac{dy}{dx} + p(x)y = q(x)$ (with a suitable renaming of variables where necessary) and identify the functions p and q.

Practice Questions

3. (i) Find the general solution of $\dfrac{dy}{dx} + 3y = x$.

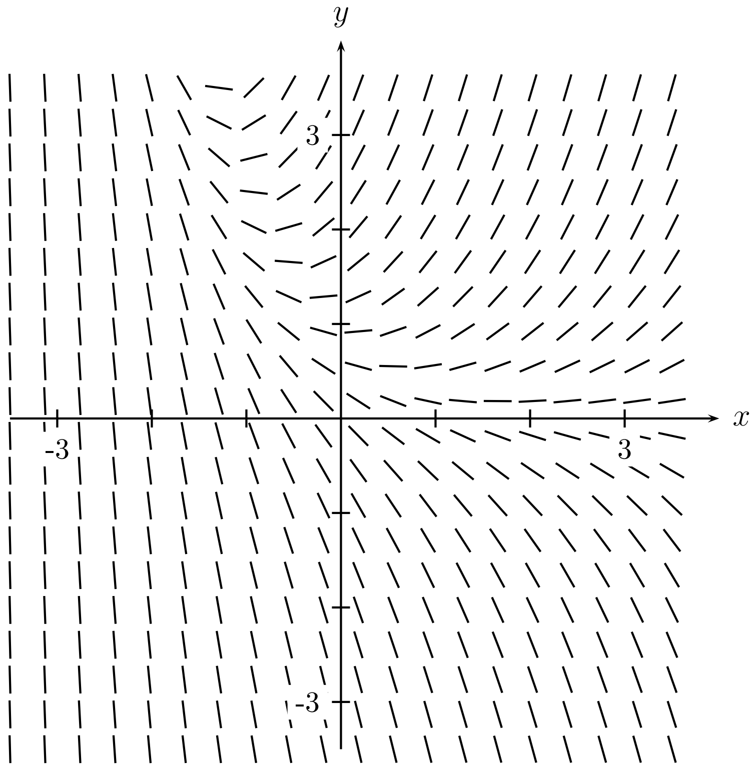
(ii) Find the general solution of $t\dfrac{dx}{dt} + x = \cos t$

(iii) Find the particular solution of $\dfrac{x}{2}\dfrac{dy}{dx} = x^2 - y$ for which $y = 1$ when $x = 1$.

4. (i) The direction field for the equation $\dfrac{dy}{dx} = y - e^{-x}$ is shown below.

Sketch the graphs of the solutions which satisfy the following initial conditions:

(a) $x = 0$, $y = 0$; (b) $x = 0$, $y = 1$.



(ii) (a) Find the general solution to the above differential equation.

(b) Find the particular solutions which correspond to each of the curves sketched in part (i). What happens as $x \to \infty$ for each of these solutions?

(c) Find the particular solution for the initial condition $x = 0$, $y = \frac{1}{2}$. What happens as $x \to \infty$ in this case?

5. (Suitable for group work and discussion.) A model for how the size of a fish varies with time is

$$\frac{dV}{dt} = -\alpha V + \beta S.$$

where V is the volume of the fish, S is its surface area, t a time variable and $\alpha > 0$ and $\beta > 0$ constants.

For particular species,

$$V = \frac{L^3}{10} \qquad \text{and} \qquad S = L^2\,,$$

and when L is measured in metres and time t in years the growth equation is

$$\frac{dV}{dt} = -V + \frac{S}{10}.$$

(i) Show that L satisfies the differential equation

$$\frac{dL}{dt} = \frac{1}{3}(1 - L)\,.$$

(ii) Solve this equation as a linear differential equation to find $L(t)$ given that $L = 0$ when $t = 0$.

(iii) What is the maximum size to which such a fish can grow?

(iv) How long does it take for such a fish to grow to 50 cm in length?

More Questions

6. (*i*) For each of the following differential equations, find the general solution and also the particular solution satisfying $y(1) = 0$.

(a) $\dfrac{dy}{dx} + 4y = e^{-2x}$

(b) $\dfrac{dy}{dx} + (\sinh x)y = (2x)e^{-\cosh x}$

(*ii*) Find the general solution of the differential equation

$$\frac{dz}{dx} + (\cot x)z = -2x\,,$$

where we assume $0 < x < \pi$.

7. Consider the equation

$$\frac{dy}{dx} + y\cos x = \cos x.$$

Solve this equation as a linear equation and then solve it as a separable equation. Are the solutions the same?

8. In electronic circuit theory, circuits with a resistor and an inductance coil in series with a voltage applied across these two components are known as RL circuits. This is because the resistance of the resistor is conventionally given as R ohms and the inductance of the coil is conventionally given as L henries. The equation for the rate of change of the electric current I in such a circuit is

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{V}{L}$$

where V is the voltage applied to the circuit. In a circuit with an applied AC current, V will vary with time as $V = A\sin\omega t$. So, if R and L are constant the equation becomes

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{A\sin\omega t}{L}.$$

Solve this equation to find the general solution for I as a function of t. Find the particular solution if the circuit has no current in it when it is switched on. What happens to the current as $t \to \infty$? How does the initial condition affect this long-term behaviour?

(Hint: $\int e^{au}\sin bu\,du = \frac{e^{au}}{a^2+b^2}(a\sin bu - b\cos bu) + C$.)

Answers to Selected Questions

1. Equations (*i*), (*ii*) and (*iii*) are linear, (*iv*) is separable.

2. (*i*) The equation is in standard form, with $p(x) = 3$, and $q(x) = x$.

(*ii*) Standard form $\dfrac{dx}{dt} + \dfrac{x}{t} = \dfrac{\cos t}{t}$. So solution is $p(t) = \dfrac{1}{t}$ and $q(t) = \dfrac{\cos t}{t}$.

(*iii*) Standard form $\dfrac{dy}{dx} + \dfrac{2y}{x} = 2x$. Solution $p(x) = \dfrac{2}{x}$ and $q(x) = 2x$.

3. (i) $y = \frac{1}{3}x - \frac{1}{9} + Ce^{-3x}$ (ii) $x = \dfrac{\sin t}{t} + \dfrac{C}{t}$ (iii) $y = \dfrac{x^2}{2} + \dfrac{1}{2x^2}$

4. (ii) (a) $y = \frac{1}{2}e^{-x} + Ce^{x}$

(b) For $y = 1$, $x = 0$ particular solution is $y = \frac{1}{2}e^{-x} + \frac{1}{2}e^{x} = \cosh x$.

For $y = 0$, $x = 0$ particular solution is $y = \frac{1}{2}e^{-x} - \frac{1}{2}e^{x} = -\sinh x$.

5. (i) $L = 1 - e^{-t/3}$ (ii) 1 metre (iii) $t = 3\ln 2 = 2.08$ years

6. (i) (a) General solution: $y = \frac{1}{2}e^{-2x} + Ce^{-4x}$, particular solution: $y = \frac{1}{2}e^{-2x}\left(1 - e^{2(1-x)}\right)$.

(b) General solution: $y = (x^2 + C)e^{-\cosh x}$, particular solution: $y = (x^2 - 1)e^{-\cosh x}$.

(ii) $z = \dfrac{2x\cos x - 2\sin x + C}{\sin x}$

8. General solution:

$$I = \frac{A}{L}\frac{1}{(R/L)^2 + \omega^2}\left(\frac{R}{L}\sin\omega t - \omega\cos\omega t\right) + Ce^{-Rt/L}$$

Particular solution:

$$I = \frac{A}{L}\frac{1}{(R/L)^2 + \omega^2}\left(\frac{R}{L}\sin\omega t - \omega\cos\omega t + \omega e^{-Rt/L}\right).$$

As $t \to \infty$

$$I \to \frac{A}{L}\frac{1}{(R/L)^2 + \omega^2}\left(\frac{R}{L}\sin\omega t - \omega\cos\omega t\right).$$