THE UNIVERSITY OF SYDNEY
MATH1003 INTEGRAL CALCULUS AND MODELLING

Semester 2	**Exercises for Week 12**	2011

Assumed Knowledge Finding the roots of quadratic equations.

Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$.

Objectives

(11a) To be able to write down the auxiliary (characteristic) equation associated with a second-order differential equation with constant coefficients.

(11b) To be able to construct the solutions to such differential equations in terms of real exponential and trigonometric functions.

Preparatory Questions

1. Write down the auxiliary (characteristic) equations for each the following second-order linear differential equations with constant coefficients, and find their roots:

(i) $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} - 8y = 0$ (ii) $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} - 4y = 0$

(iii) $\dfrac{d^2y}{dt^2} - 9y = 0$ (iv) $\dfrac{d^2y}{dt^2} - 2\dfrac{dy}{dt} + 5y = 0$

(v) $\dfrac{d^2x}{dt^2} + 2\dfrac{dx}{dt} + x = 0$

Practice Questions

2. Find the particular solution of Preparatory Question 1 (i) with $y(0) = 0$ and $y'(0) = 3$.

3. Find the general solution of Preparatory Question 1 (ii).

4. Find the particular solution of Preparatory Question 1 (iii) which satisfies the initial conditions $y = 3$ and $\dfrac{dy}{dt} = 3$ when $t = 0$.

5. Find the general solution of Preparatory Question 1 (iv).
Express your answer in terms of real functions.
What is the particular solution satisfying $y(0) = 1$ and $y(\pi/4) = 2$?

6. Find the particular solution of Preparatory Question 1 (v) which satisfies the initial conditions $x(0) = 1$ and $x'(0) = 2$.

More Questions

7. Find the general solutions of these second-order homogeneous equations.
In each case give your answer in terms of real functions.

(i) $2\dfrac{d^2y}{dx^2} - 7\dfrac{dy}{dx} + 3y = 0.$ (ii) $\dfrac{d^2y}{dx^2} + 3y = 0.$

(iii) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = 0.$

8. Find the general solution of each of these second-order linear homogeneous equations.
Hence find the particular solution for the given conditions.
In each case give your answer in terms of real functions.

(i) $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 20y = 0, \quad y(0) = y'(0) = 1$

(ii) $\dfrac{d^2y}{dx^2} + 9y = 0, \quad y(0) = 1, \quad y(\pi/6) = 3.$

(iii) $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 7y = 0, \quad y(0) = 0, \quad \dfrac{dy}{dt} = 3$ when $t = 0.$

(iv) $\dfrac{d^2x}{dt^2} - 4\dfrac{dx}{dt} + 4x = 0, \quad x(0) = 1, \quad x(1) = 3e^2.$

Answers to Selected Questions

1. (i) The auxiliary equation $\lambda^2 + 2\lambda - 8 = 0$. Roots $\lambda = 2$ or $\lambda = -4$.
(ii) The auxiliary equation $\lambda^2 + 2\lambda - 4 = 0$. Roots $\lambda = -1 - \sqrt{5}$.
(iii) The auxiliary equation $\lambda^2 - 9 = 0$. Roots $\lambda = 3$ or $\lambda = -3$.
(iv) The auxiliary equation $\lambda^2 - 2\lambda + 5 = 0$. Roots $\lambda = 1 \pm 2i$.
(v) The auxiliary equation $\lambda^2 + 2\lambda + 1 = 0$. This has two equal roots $\lambda = -1$.

2. $y = \frac{1}{2}e^{2t} - \frac{1}{2}e^{-4t}$

3. $y = e^{-t}(Ae^{\sqrt{5}t} + Be^{-\sqrt{5}t})$

4. $y = 2e^{3t} + e^{-3t}$

5. General solution, real form, is $y = e^t(E\cos 2t + F\sin 2t)(1 + 2i)t + Be^{(1-2i)t}$.
Particular solution is $y = e^t(\cos 2t + 2e^{-\pi/4}\sin 2t)$

6. $x = 3te^{-t} + e^{-t}$

7. (i) $y = Ae^{x/2} + Be^{3x}$ (ii) $y = A\cos\sqrt{3}x + B\sin\sqrt{3}x$
(iii) $y = e^x(A\cos x + B\sin x)$

8. (i) General solution $y = Ae^{-5x} + Be^{4x}$. Particular solution $y = \dfrac{1}{3}e^{-5x} + \dfrac{2}{3}e^{4x}$
(ii) General solution $y = E\cos 3x + F\sin 3x$. Particular solution $y = \cos 3x + 3\sin 3x$.
(iii) General solution $y = e^{-2t}(E\cos\sqrt{3}t + F\sin\sqrt{3}t)$. Particular solution $y = \sqrt{3}e^{-2t}\sin\sqrt{3}t$.
(iv) General solution $x = Ae^{2t} + Bte^{2t}$. Particular solution $x = e^{2t} + 2te^{2t}$.